

## Part II Algebraic Topology – Example Sheet 4

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Questions marked \* are optional and should only be attempted after non-starred questions.

1. For each of the following exact sequences of abelian groups and homomorphisms say as much as possible about the unknown group  $A$  and homomorphism  $\alpha$ .

- (i)  $0 \longrightarrow \mathbb{Z}/2 \longrightarrow A \longrightarrow \mathbb{Z} \longrightarrow 0$ ,
- (ii)  $0 \longrightarrow \mathbb{Z}/2 \longrightarrow A \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ ,
- (iii)  $0 \longrightarrow A \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ ,
- (iv)  $0 \longrightarrow \mathbb{Z}/3 \longrightarrow A \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$ .

2. Consider a commutative diagram

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \downarrow h_5 \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

in which the rows are exact and each square commutes. If  $h_1, h_2, h_4$ , and  $h_5$  are isomorphisms, show that  $h_3$  is too.

3. Let  $K$  be a simplicial complex in  $\mathbb{R}^m$ , and consider this as lying inside  $\mathbb{R}^{m+1}$  as the vectors of the form  $(x_1, \dots, x_n, 0)$ . Let  $e_+ = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$  and  $e_- = (0, \dots, 0, -1) \in \mathbb{R}^{m+1}$ . The *suspension* of  $K$  is the simplicial complex in  $\mathbb{R}^{m+1}$

$$SK := K \cup \{ \langle v_0, \dots, v_n, e_+ \rangle, \langle v_0, \dots, v_n, e_- \rangle \mid \langle v_0, \dots, v_n \rangle \in K \}.$$

- (a) Show that  $SK$  is a simplicial complex, and that if  $|K| \cong S^n$  then  $|SK| \cong S^{n+1}$ .
  - (b) Using the Mayer–Vietoris sequence, show that if  $K$  is connected then  $H_0(SK) \cong \mathbb{Z}$ ,  $H_1(SK) = 0$ , and  $H_i(SK) \cong H_{i-1}(K)$  if  $i \geq 2$ .
  - (c) If  $f : K \rightarrow K$  is a simplicial map, let  $Sf : SK \rightarrow SK$  be the unique simplicial map which agrees with  $f$  on the subcomplex  $K$  and fixes the points  $e_+$  and  $e_-$ . Show that under the isomorphism in (ii), the maps  $f_*$  and  $Sf_*$  agree. [Hint: It may help to describe the isomorphism in (ii) at the level of chains.]
  - (d) Deduce that for every  $n \geq 1$  and  $d \in \mathbb{Z}$  there is a map  $f : S^n \rightarrow S^n$  so that  $f_*$  induces multiplication by  $d$  on  $H_n(S^n) \cong \mathbb{Z}$ .
4. If  $K$  is a simplicial complex with  $H_i(K) \cong \mathbb{Z}^r \oplus F$ , for  $F$  a finite abelian group, show that  $H_i(K; \mathbb{Q}) \cong \mathbb{Q}^r$ . [Hint: There is a chain map  $C_\bullet(K) \rightarrow C_\bullet(K; \mathbb{Q})$ .]
5. By describing a triangulation of  $S^n$  which is preserved under the antipodal map, show that  $\mathbb{RP}^n$  has a triangulation. [Be careful that the triangulation you describe actually comes from a simplicial complex! Some subdivision may be necessary.] Using the Mayer–Vietoris sequence, show that there is an exact sequence

$$0 \longrightarrow H_n(\mathbb{RP}^n) \longrightarrow \mathbb{Z} \longrightarrow H_{n-1}(\mathbb{RP}^{n-1}) \longrightarrow H_{n-1}(\mathbb{RP}^n) \longrightarrow 0$$

and that  $H_i(\mathbb{RP}^{n-1}) \rightarrow H_i(\mathbb{RP}^n)$  is an isomorphism for  $i < n-1$ . Hence show that

$$H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or if } i = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } i \text{ is odd and } 0 < i < n \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that  $\mathbb{RP}^{2k}$  does not retract onto  $\mathbb{RP}^{2k-1}$ , and that any map  $f : \mathbb{RP}^{2k} \rightarrow \mathbb{RP}^{2k}$  has a fixed point.

6. Let  $A$  be a  $2 \times 2$  matrix with entries in  $\mathbb{Z}$ . Show that the linear map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  preserves the equivalence relation  $(a, b) \sim (a', b') \iff (a - a', b - b') \in \mathbb{Z}^2$ , and so induces a continuous map  $f_A$  from the torus  $T$  to itself. Compute the effect of the continuous map  $f_A$  on the homology of  $T$ .
7. For triangulated surfaces  $X$  and  $Y$ , let  $X \# Y$  be the surface obtained by cutting out a 2-simplex from both  $X$  and  $Y$  and then gluing together the two copies of  $\partial\Delta^2$  formed. Use the Mayer–Vietoris sequence to compute the homology of  $\Sigma_g \# S_n$ , and hence deduce that it is homeomorphic to  $S_{n+2g}$ .
8. Let  $p : \tilde{X} \rightarrow X$  be a finite-sheeted covering space, and  $h : |K| \rightarrow X$  a triangulation. Show that there is an  $r \geq 1$  and triangulation  $g : |L| \rightarrow \tilde{X}$  so that the composition  $h^{-1} \circ p \circ g : |L| \rightarrow |K^{(r)}|$  is a simplicial map. If  $p$  has  $n$  sheets, show that  $\chi(\tilde{X}) = n \cdot \chi(X)$ . Hence show that  $\Sigma_g$  is a covering space of  $\Sigma_h$  if and only if  $\frac{1-g}{1-h}$  is an integer. *[Hint: If  $g = 1 + k \cdot (h - 1)$ , show that  $\mathbb{Z}/k$  acts freely and properly discontinuously on a particular orientable surface of genus  $g$ , and identify the quotient.]*
9. Let  $p : S^{2k} \rightarrow X$  be a covering map,  $G = \pi_1(X, [x_0])$ , and recall that  $G$  then acts freely on  $S^{2k}$ . Show that for any  $g \in G \setminus \{e\}$  the map  $g_* : H_{2k}(S^{2k}) \rightarrow H_{2k}(S^{2k})$  is multiplication by  $-1$ . Deduce that  $G$  is either trivial or  $\mathbb{Z}/2$ , and that  $\mathbb{RP}^{2k}$  is not a proper covering space of any other space.
10. If  $f : K \rightarrow K$  is a simplicial isomorphism, let  $X \subset |K|$  be the fixed set of  $|f|$  i.e.  $\{x \in |K| \mid |f|(x) = x\}$ . Show that the Lefschetz number  $L(f)$  is equal to  $\chi(X)$ . *[Hint: Barycentrically subdivide  $K$  so that  $X$  is the polyhedron of a sub simplicial complex.]*

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